UNIVERSITY OF MANITOBA
Term Test 1C
COURSE: MATH 1500
DATE & TIME: October 9, 2018, 5:40PM – 6:40PM
CRN: various
DURATION: 1 hour EXAMINER: various

INSTRUCTIONS

I. No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

II. This exam has a title page, 6 pages of questions and two blank pages for rough work. Please check that you have all the pages. **DO NOT REMOVE THE SCRAP PAPER**

III. The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 50 points.

IV. **Answer all questions on the exam paper** in the space provided beneath the question. Unjustified answers will receive little or no credit. **Do not continue on the back of the page.** If you need more space, continue on one of the scrap pages, CLEARLY INDICATING THAT YOUR WORK IS TO BE CONTINUED.

V. Do not deface the QR - code in the top right corner. Doing so may result in the page not being scanned and therefore not graded.

Question	Points	Score
1	20	
2	7	
3	7	
4	5	
5	5	
6	6	
Total:	50	

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1. Calculate each limit below, if it exists. If a limit does not exist, explain why. Show all work. Writing an answer with no justification may not yield any marks.

[3] (a) $\displaystyle\lim_{x\to 2^-} \frac{3-x^2}{|x-2|}$.

Solution: $\displaystyle\lim_{x\to 2^-} 3-x^2 = -1$ and $\displaystyle\lim_{x\to 2^-} |x-2| = \lim_{x\to 2^-} -(x-2) = 0^+$.
Therefore the limit goes to either $\pm\infty$.
Since the fraction is negative, $\displaystyle\lim_{x\to 2^-} f(x) = -\infty$.

[5] (b) $\displaystyle\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-\sqrt{2x}}$.

Solution: We rationalize the denominator:

$$\begin{aligned}\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-\sqrt{2x}} &= \lim_{x\to 2} \frac{(x-2)(\sqrt{x+2}+\sqrt{2x})}{(\sqrt{x+2}-\sqrt{2x})(\sqrt{x+2}+\sqrt{2x})}\\ &= \lim_{x\to 2} \frac{(x-2)(\sqrt{x+2}+\sqrt{2x})}{x+2-2x}\\ &= \lim_{x\to 2} \frac{(x-2)(\sqrt{x+2}+\sqrt{2x})}{-(x-2)}\\ &= \lim_{x\to 2} -(\sqrt{x+2}+\sqrt{2x})\\ &= -4.\end{aligned}$$

[6] (c) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{4x^2+1}}{5x+2}$.

Solution: Notice that $\sqrt{x^2} = -x$ because $x < 0$.

$$\begin{aligned}\lim_{x\to -\infty} \frac{\sqrt{4x^2+1}}{5x+2} &= \lim_{x\to -\infty} frac\sqrt{x^2(4+\frac{1}{x^2})}5x+2 = \lim_{x\to -\infty} \frac{\sqrt{x^2}\sqrt{(4+\frac{1}{x^2})}}{x(5+\frac{2}{x})}\\ &= \lim_{x\to -\infty} \frac{-x\sqrt{(4+\frac{1}{x^2})}}{x(5+\frac{2}{x})} = \lim_{x\to -\infty} \frac{-\sqrt{(4+\frac{1}{x^2})}}{(5+\frac{2}{x})}\\ &= \frac{-\sqrt{(4+0)}}{(5+0)} = -\frac{2}{5}.\end{aligned}$$

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[2] (d) $\lim_{x\to 5} \frac{x^2 - 4x - 5}{x^2 - 25}$.

Solution:

$$\lim_{x\to 5} \frac{x^2 - 4x - 5}{x^2 - 25} = \lim_{x\to 5} \frac{(x-5)(x+1)}{(x+5)(x-5)} = \lim_{x\to 5} \frac{(x+1)}{(x+5)} = \frac{3}{5}$$

[4] (e) $\lim_{x\to 3}(x-3)^2 \sin\left(\frac{\pi}{(x-3)^6}\right)$.

Solution: For all $x \neq 3$, $-1 \leq \sin\left(\frac{\pi}{(x-3)^6}\right) \leq 1$.

Thus, for $x \neq 3$,

$$-(x-3)^2 \leq (x-3)^2 \sin\left(\frac{\pi}{(x-3)^6}\right) \leq (x-3)^2.$$

Since both $(x-3)^2 \to 0$ and $-(x-3)^2 \to 0$, as $x \to 3$, then by the Squeeze Theorem,

$$\lim_{x\to 3}(x-3)^2 \sin\left(\frac{\pi}{(x-3)^6}\right) = 0.$$

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[7] 2. Let f be the function:

$$f(x) = \begin{cases} kx+7 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ k^2x-5 & \text{if } x > 2 \end{cases}.$$

Find all values of k for which $f(x)$ is continuous for all real numbers. Be sure to fully justify your answer.

Solution: For any k, the function is continuous on $(-\infty, 2) \cup (2, \infty)$ because it is a polynomial in those intervals.
The function will be continuous at $x = 2$ if

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = 3.$$

Furthermore,

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} kx+7 \\ &= 2k+7. \end{aligned} \qquad \begin{aligned} \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} k^2x-5 \\ &= 2k^2-5. \end{aligned}$$

Then,

$$2k+7 = 3 = 2k^2-5.$$

We now find k:

$$\begin{aligned} 2k+7 &= 3 \\ 2k &= -4 \\ k &= -2 \end{aligned} \qquad \begin{aligned} 2k^2-5 &= 3 \\ k &= \pm\sqrt{4} \\ k &= \pm 2. \end{aligned}$$

The only value of k that makes both above statements true is $k = -2$. We conclude that for $k = -2$ the function $f(x)$ is continuous for all real numbers.

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[7] 3. Use the **definition of derivative** to find $f'(x)$. No credit will be given for any other method.

$$f(x) = \frac{1}{5-x}$$

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{1}{5-(x+h)} - \dfrac{1}{5-x}}{h} \\ &= \lim_{h\to 0} \frac{5-x-(5-(x+h))}{h(5-(x+h))(5-x)} \\ &= \lim_{h\to 0} \frac{5-x-5+x+h}{h(5-x-h)(5-x)} \\ &= \lim_{h\to 0} \frac{h}{h(5-x-h)(5-x)} \\ &= \lim_{h\to 0} \frac{1}{(5-x-h)(5-x)} \\ &= \frac{1}{(5-x)^2}. \end{aligned}$$

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[5] 4. Find the domain of the function $f(x) = \sqrt{\dfrac{(x^2-9)}{(x-7)}}$. Express your final answer in terms of intervals.

Solution: $f(x) = \sqrt{\dfrac{(x^2-9)}{(x-7)}} = \sqrt{\dfrac{(x-3)(x+3)}{(x-7)}}$.
The radicand needs to be non-negative under the square root and non-zero in the denominator. Let $h(x) = \dfrac{(x-3)(x+3)}{(x-7)}$

x	$x < -3$	$-3 < x < 3$	$3 < x < 7$	$x > 7$
$x-3$	$-$	$-$	$+$	$+$
$x+3$	$-$	$+$	$+$	$+$
$x-7$	$-$	$-$	$-$	$+$
$h(x)$	$-$	$+$	$-$	$+$

Therefore, $D_f = [-3, 3] \cup (7, \infty)$.

[5] 5. Use the Intermediate Value Theorem to show that $x^3 + x^2 - \dfrac{1}{x-2} = 0$ has at least one solution on the interval $[-2, 1]$.

Solution: Let $f(x) = x^3 + x^2 - \dfrac{1}{x-2}$, then $f(x)$ is continuous for all x except $x = 2$.
In particular, $f(x)$ is continuous on the interval $[-2, 1]$.
We now check the value of $f(x)$ at the end points:

$$f(-2) = (-2)^3 + (-2)^2 - \frac{1}{-2-2} = -\frac{15}{4} < 0, \quad f(1) = (1)^3 + (1)^2 - \frac{1}{1-2} = 3 > 0$$

Since the values of the function at the end points have different signs, using the Intermediate Value Theorem, we conclude that there is a point c on the interval $[-2, 1]$ such that $f(c) = 0$. Therefore, there exist at least one solution of the equation on $[-2, 1]$.

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[6] 6. Use the **definition of derivative** to show that

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 1 \\ -2x + 4 & \text{if } x > 1 \end{cases}$$

is not differentiable at $x = 1$.

Solution:

The right-hand derivative of f at $x = 1$ is given by

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[-2(1+h) + 4] - 2}{h} = \lim_{h \to 0^+} \frac{-2h}{h} = -2.$$

The left-hand derivative of f at $x = 1$ is given by

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[(1+h)^2 + 1] - 2}{h} = \lim_{h \to 0^-} (2 + h) = 2.$$

Since the left-hand and right-hand derivatives differ at $x = 1$, the function f is not differentiable there.

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